DS 1:
Random Walks and the Mean Free Path

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1 The Stirling approximation

Factorials appear very often in physics (especially in Statistical Mechanics) but they can be quite difficult to work with since they very quickly become unmanageably large. One interesting result was discovered by Euler who noticed that for integers n,

$$\int_0^\infty \mathrm{d}u\, u^n e^{-u} = n! \quad (n = 1, 2, 3, \ldots).$$

The above integral is not only defined for integers. In general, one could replace n by some continuous variable z, and the resulting integral is called the Gamma Function, denoted by the Greek symbol Γ. You will learn more about this function in later courses on Mathematical Physics:

$$\int_0^\infty \mathrm{d}u\, u^z e^{-u} = \Gamma(z+1).$$

In this problem, we will deal with approximating the factorial using through the Gamma Function, and we will do it using both Gaussian Integration and the Taylor Series, two techniques you should already be familiar with.

(a) Start off by looking at the function that's being integrated. Show by plotting or sketching the functions u^n and e^{-u} what their product looks like, and explain why you'd expect it to have a maximum.

(b) Next, show that we can write the integral as

$$n! = \int_0^\infty \mathrm{d}u\, e^{-(u - n\log u)}.$$

We will approximate this integral as follows: The function in the integral decays exponentially fast, so most of the area under it will be very close to its maximum. Convince yourself that the maximum of the exponential occurs when the function $u - n\log u$ is minimum.

Show that the function $f(u) = u - n\log u$ has a *minimum* when $u = n$.

(c) If we do a Taylor Expansion of $f(u)$ about this minimum, the first non-trivial term will then be quadratic (why?). Show that about this minimum:

$$f(u) = (n - n\log n) + \frac{(u-n)^2}{2n} + \ldots$$

Show that this implies that we have approximated the integrand as a Gaussian.

(d) Use this to show that you can approximate

$$n! \approx n^n e^{-n} \int_0^\infty \mathrm{d}u \exp\left(-\frac{(u-n)^2}{2n}\right)\Big\{1+\dots\Big\}.$$

(e) Now, the integral above *nearly* looks like a Gaussian integral, except for the limits. However, it turns out that if we change the integral's lower limit from 0 to $-\infty$, the error introduced is tiny. We will now show this numerically.

Coding Exercise: Integrate the following expressions numerically to find the functions below and plot them on the same graph for some range of (integers) n:

$$I_1(n) = n! \qquad I_2(n) = n^n e^{-n} \int_0^\infty \mathrm{d}u \exp\left(-\frac{(u-n)^2}{2n}\right) \qquad I_3(n) = n^n e^{-n} \int_{-\infty}^\infty \mathrm{d}u \exp\left(-\frac{(u-n)^2}{2n}\right).$$

Next, plot the relative error associated with the functions $I_2(n)$ and $I_3(n)$, i.e.

$$\frac{\Delta I_2}{I_1} = \frac{I_1(n) - I_2(n)}{I_1(n)} \qquad \frac{\Delta I_3}{I_1} = \frac{I_1(n) - I_3(n)}{I_1(n)}.$$

Beyond which value of n does their difference become less than 1%?

(f) Show that we can now approximate[1]

$$n! \approx n^n e^{-n} \int_{-\infty}^\infty \mathrm{d}u \exp\left(-\frac{(u-n)^2}{2n}\right) = n^n e^{-n}\sqrt{2\pi n}\ \Big\{1+\dots\Big\}.$$

2 Kinetic theory: mean-free paths

Kinetic theory has applications in many places in Physics. Let us use the different ideas we saw this week to address some very specific questions. From class, we saw that the mean free path depends not only on the number density of the medium n, but also on its interaction cross-section σ, i.e.

$$\lambda = \frac{1}{n\sigma}. \tag{1}$$

Let us now ask ourselves the following deceptively simple question: How long does it take for a photon created at the centre of the sun to reach you?

(a) Start by doing a naive calculation assuming that the photon doesn't interact with anything.

(b) In reality, the photon interacts with free electrons by scattering off them in arbitrary directions. This is called *Thompson scattering*, and this process has a cross-section of

$$\sigma_T = \frac{8\pi}{3}\left(\frac{e^2}{m_e c^2}\right)^2 \approx 6.7\times10^{-25}\ \mathrm{cm}^2. \tag{2}$$

[1] In physics, it is often sufficient to ignore the $\sqrt{2\pi n}$ term, so that the Stirling Approximation is often simply written as: $n! \approx n^n e^{-n}$. Here is a much quicker proof of that identity, where all we do is replace a sum by an integral:

$$\log(n!) = \sum_{i=1}^n \log i \approx \int_1^n \log x \mathrm{d}x = n\log n - n \qquad \Longrightarrow \qquad n! = n^n e^{-n}.$$

However, the method described in the exercise is better, as it provides a systematic way to find higher order correction terms.

(c) Assuming all the gas is hydrogen of mass m_H, and that there is one electron per atom, show that the number of electrons in the star is

$$n_e \approx \frac{\rho}{m_H}. \tag{3}$$

(d) Use this to show that the mean free path for electron scattering is

$$\lambda_{\text{es}} \approx 2 \text{ cm}. \tag{4}$$

You may assume that the average density of the sun is 1.4 g/cc.

(e) Now, convince yourself that the photon is executing a random walk when moving from the centre to the surface of the Sun. We have seen that for a random walk of N steps of size l, the mean-squared displacement R satisfies

$$\langle R_d^2 \rangle = N l^2. \tag{5}$$

How many steps does it need to take to get to the surface of the Sun, if $l = \lambda_{\text{es}}$?

(f) If you assume that it moves at c between scatterings, how long will it take to travel between the centre and the surface of the Sun? (In reality, the scattering is not dominated by electrons, and the mean-free path is closer to $\lambda = 1$ mm. What effect would this have on the time?)