DS 2: The Microcanonical Ensemble

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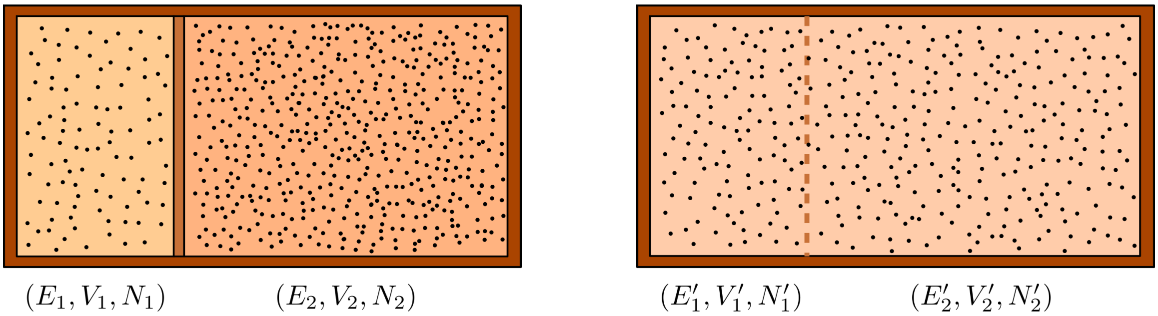
1 The classical ideal gas

Consider an ideal classical gas in an isolated box. The box is initially partitioned into two unequal compartments, isolated from each other, as shown in Figure (1a). The energy, volume, and number of particles in each of the compartments is initially different. The separating wall is then suddenly made porous, allowing the transfer of energy and matter between the compartments. In this question, we will try to understand what we can say of the final configuration, once equilibrium has been reached.



(a) Initial configuration: separated compartments.

(b) Final configuration: after allowing for mixing.

Figure 1: The initial (left) and final (right) configurations of the system, denoted by the unprimed and primed values respectively. The compartments are allowed to transfer particles, and thus energy, between each other. How are the final energies, volumes, and numbers in each compartment related to the initial values?

(a) First consider an ideal gas of N particles in three dimensions with total energy E and volume V. Show, as we did in the last lecture, that the entropy of such a system is given by the Sackur-Tetrode equation:

$$S(E,V,N) = N\log\left[\frac{V}{N}\left(\frac{4\pi mE}{3h^2N}\right)^{3/2}\right] + \frac{5}{2}N \tag{1}$$

(b) Argue that we can write the total entropy of the final configuration of the gas as

$$S(E,V,N) = S(E_1,V_1,N_1) + S(E_2,V_2,N_2). \tag{2}$$

(c) At equilibrium, the entropy will be maximised with respect to all the variables (E_1, V_1, N_1) and (E_2, V_2, N_2), subject to the constraint that the total energy always be E, and total number of particles by N. We can incorporate this constraint quite simply by using the technique of Lagrange Multipliers, as discussed in the lecture. Define a new function

$$\mathcal{F}(E_1, V_1, N_1, E_2, V_2, N_2, \lambda_E, \lambda_M) = S(E_1, V_1, N_1, E_2, V_2, N_2) + \lambda_E(E_1 + E_2 - E) - \lambda_N(N_1 + N_2 - N), \quad (3)$$

where λ_E and λ_M are the two Lagrange multipliers. Now, we can maximise $\mathcal{F}$ while treating all the parameters as independent, and ignoring the constraints at the expense of introducing these new parameters. Minimise $\mathcal{F}$ with respect to E_1 and E_2 and thus show that at equilibrium

$$\frac{E_1}{N_1} = \frac{E_2}{N_2}. \quad (4)$$

What does this mean, physically?

(d) Next, maximise $\mathcal{F}$ with respect to N_1 and N_2, and thus show that at equilibrium

$$\frac{N_1}{V_1} = \frac{N_2}{V_2}. \quad (5)$$

What does *this* mean, physically?

(e) What do the last two results tell us about the energy *densities* in the two boxes. How are they related?