DS 4:
The Canonical Ensemble

Philip Cherian
September 29, 2023

1 The thermal de Broglie wavelength

(a) Consider an ideal monoatomic gas of N particles in 3-dimensions. Derive the Sackur-Tetrode for its entropy.

$$S(E,V,N) = N\log\left[\frac{V}{N}\left(\frac{4\pi mE}{3h^2N}\right)^{3/2}\right] + \frac{5}{2}N \tag{1}$$

(b) Set $T \to 0$, and argue that this equation leads to a result that goes against your physical intuition.

(c) Rewrite the above equation in terms of a new quantity λ_{th}, which we will call the *thermal* de Broglie wavelength

$$\lambda_{\text{th}} = \frac{h}{\sqrt{2\pi mT}}. \tag{2}$$

If h is Planck's constant, what does this wavelength correspond to? Argue that it can be thought of as a length-scale associated with quantum-effects.

(d) Use the above result to argue that the Sackur-Tetrode equation is only valid for a classical ideal gas when

$$\frac{V}{N\lambda_{\text{th}}^3} \gg 1, \tag{3}$$

and so the limit taken in part (b) was unphysical in the first place.

2 The ultra-relativistic gas revisited

(a) Consider first a *non-relativistic* gas of N particles in 3 dimensions, and compute the *adiabatic index* γ. Show that

$$\gamma_{\text{non-rel}} = \frac{5}{3}. \tag{4}$$

(b) Next, consider the same gas to be *ultra*-relativistic, and compute $\gamma_{\text{ultra-rel}}$. Is it the same as $\gamma_{\text{non-rel}}$?

(c) What is the "thermal de Broglie wavelength" for an ultra-relativistic gas? Show that

$$\lambda_{\text{th}}^{\text{rel}} = \frac{hc}{2\pi^{1/3}T}. \tag{5}$$

3 The relativistic gas

Consider a two-dimensional gas of massive, non-interacting, relativistic particles with energies

$$\mathcal{H} = \sqrt{\mathbf{p}^2 c^2 + m^2 c^4}. \tag{6}$$

We will solve this problem now without making any approximations.

(a) Compute the partition function for this system.

(b) Compute the free-energy $F(T, V, N)$.

(c) Compute the entropy $S(T, V, N)$.

(d) Find the equation of state of such a gas.

4 The Maxwell-Boltzmann distribution

We now try to ask ourselves the following question in light of the canonical ensemble: what is the probability that a randomly chosen particle of a classical ideal gas (in a box of volume V and with energy E) has a momentum between $\mathbf{p}$ and $\mathbf{p} + \mathrm{d}\mathbf{p}$? Clearly, using our current formalism, this is just

$$\mathcal{P}(\mathbf{p})\mathrm{d}^3\mathbf{p} = \frac{\text{Number of microstates corresponding to this specific particle having a momentum in this range}}{\text{Total number of microstates given this energy and volume}}. \tag{7}$$

Use this to compute the Maxwell-Boltzman *speed* distribution in d dimensions.