DS 9: The Density of States

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1 Density of states in $d-$dimensions

(a) Consider a gas of fermions (say, electrons) in a $d-$dimensional box, and compute the momentum-space density of states, $g(k)\mathrm{d}k$.

(b) First consider the case of non-relativistic matter. Using the dispersion relation

$$\epsilon = \frac{p^2}{2m} = \frac{\hbar^2 k^2}{2m}, \tag{1}$$

compute the density of states in energy $g(\epsilon)$.

Hint: You will need to use the fact that in $d-$dimensions

$$\mathrm{d}^d k = S_d^{(1)} k^{d-1} = \frac{2\pi^{d/2}}{\Gamma(d/2)} k^{d-1}, \tag{2}$$

where $S_d^{(1)}$ is the surface area of a unit sphere in d dimensions.

(c) Find the Fermi energy ϵ_F in terms of the total number of fermions N.

(d) Use ϵ_F to compute the internal energy E when $T \approx 0$.

(e) From the variation of the internal energy with volume, compute the *pressure* of this system. This pressure, due solely to the fact that these particles are fermions, is thus known as the "degeneracy pressure".

(f) Now, repeat the above calculations for a gas of fermions that are *relativistic*, using the dispersion relation for relativistic particles,

$$\epsilon = pc = \hbar k c. \tag{3}$$

2 White Dwarfs in $d-$dimensions

(a) Consider a star being held at equilibrium at some radius R. The gravitational potential energy of such a system in 3-dimensions is given by

$$E_g = -\int_0^R \frac{GM(r) \times 4\pi r^2 \rho(r)}{r} \mathrm{d}r, \tag{4}$$

where $\rho(r)$ is the density at every point r, $M(r)$ is the mass contained in a radius r, and G is the gravitational constant. What would the corresponding potential energy be in d dimensions?

(b) If we assume the density of this system to be a constant ρ, compute the integral for the gravitational energy in d dimensions.

(c) Use this number to compute the gravitational *pressure* in d dimensions.

(d) In a very crude sense, White Dwarfs are stable because they can find a radius at which their gravitational pressures and degeneracy pressures can equalise. Are White Dwarfs stable in all dimensions?