DS 11: Mean Field Solution of the Ising Model

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1 Recap: the non-interacting spin lattice

Consider an array of N spins in a cubic lattice in $d-$dimensions. Each spin s_i can have value ± 1. All the spins interact with an external magnetic field and with their nearest neighbours, so that the energy of the system is

$$\mathcal{H} = -J \sum_{\langle i,j \rangle} s_i s_j - h \sum_{i=1}^{N} s_i. \tag{1}$$

(a) To begin, let us neglect the inter-particle interaction, which boils down to setting $J = 0$. In this case, calculate the partition function, and thus the expected value of the average magnetisation per site $m = M/N$.

(b) Plot m as a function of h (both negative and positive values of h), for two values of T. Mark your axes and label your curves so that all the important information is evident.

2 The zeroth approximation of the Ising model

Solving the complete problem with $J \neq 0$ has only been done analytically in 1D and 2D. Not analytical solutions exist yet for 3D and higher. Here, we will apply an interesting approximation that greatly simplifies the the problem, and allows us to extract some very important information from the system without solving it exactly.

To do this, we use the "mean-field" approximation, in which we assume that each spin s_i effectively interacts with an *average* magnetisation due to all the other spins. As a result, we are effectively ignoring local variations (fluctuations) in the magnetisation. To incorporate this, we write each spin as

$$s_i = m + \Delta s_i, \tag{2}$$

where m is the average magnetisation per spin, and Δs_i is the fluctuation in the ith spin about the magnetisation m. This quantity is assumed to be very small, i.e. $\Delta s_i \ll 1$.

(a) Allowing for $J \neq 0$, use the mean field approximation to show that the energy of the system can be written as

$$\mathcal{H} = +Jm^2 \left(\sum_{\langle i,j \rangle} 1 \right) - 2Jm \left(\sum_{\langle i,j \rangle} s_i \right) - Nhm. \tag{3}$$

(b) Convince yourselves that the sum over nearest neighbours can be written as

$$\sum_{\langle i,j \rangle} = \frac{1}{2} \sum_i \sum_{\text{nn}(i)} = \frac{q}{2} \sum_i, \tag{4}$$

where $\sum_{\mathrm{nn}(i)}$ is the sum of nearest-neighbours of s_i, and q is the number of nearest-neighbours for any one spin. Use this to show that

$$\mathcal{H} = \frac{JqNm^2}{2} - (Jqm + h)\sum_i s_i. \tag{5}$$

What is interesting about this form of the energy? How does it relate to the "non-interacting" case?

(c) Determine the partition function. Now, set $h = 0$ and determine the magnetisation for this system from $\mathcal{Z}$. You should arrive at a "self-consistency" equation for m:

$$m = \tanh\left(\beta Jqm\right). \tag{6}$$

(d) Clearly, $m = 0$ is a solution to this equation. Does the above equation possess a solution where $m \neq 0$? If it does, that means that there are situations where *spontaneous* magnetisation – magnetisation in the absence of an external field – exists. This would essentially explain ferromagnetism.

(e) Plot the left- and right-hand sides of Equation (6). Show that there exists a "critical" temperature T_c, such that when $T < T_c$, the possibility of spontaneous magnetisation exists.