DS 12:
Critical Exponents of the Ising Model

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1 The Landau theory of phase transitions

In the previous DS you saw that the mean-field solution to the Ising model had a critical temperature T_c such that when $T > T_c$, only one solution exists for the magnetisation, but when $T < T_c$, three different solutions exist.

Which of these solutions is the one that the system chooses? In order to answer this question, we must look at the *free energy* of this system. The solution that minimises the free energy is the one that the system chooses.

(a) First, write out the free energy per particle $f(m)$ of this system.

(b) Convince yourselves that when $T < T_c$ but very close to T_c, m will be very small. As a result, we can expand the free energy $f(m)$ as a function of m. The tricky term in this expansion is one that will involve a function like $\log\cosh x$. Show that

$$\log(\cosh x) = -\frac{1}{2}\log\left(1-\tanh^2 x\right). \tag{1}$$

(c) Use this result to expand your free energy (when $h = 0$) and show that

$$f(m) = f(0) + \frac{(T-T_c)}{2}m^2 + \frac{T}{12}m^4, \tag{2}$$

where T_c is the critical temperature below which spontaneous magnetisation is possible.

(d) Sketch $f(m)$ for $T > T_c$ and for $T < T_c$, and explain clearly the relationship between the solutions found in the previous section in the two regimes, and the corresponding forms of $f(m)$.

(e) Find the values of the spontaneous magnetisation in this limit (i.e. when $m \ll 1$). Show that

$$m \sim (T-T_c)^{\beta}, \qquad \text{where} \qquad \beta = 1/2. \tag{3}$$

(f) Similarly, show that the susceptibility is given by

$$\chi = \left(\frac{\partial M}{\partial h}\right)_{h=0} = \quad (T-T_c)^{-\gamma} \times \begin{cases} 1, & T < T_c, \\ \frac{1}{2}, & T > T_c, \end{cases} \qquad \text{where} \qquad \gamma = 1. \tag{4}$$